

STRONG COPRIMALITY AND STRONG IRREDUCIBILITY OF ALEXANDER POLYNOMIALS

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ABSTRACT. A polynomial $f(t)$ with rational coefficients is *strongly irreducible* if $f(t^k)$ is irreducible for all positive integers k . Likewise, two polynomials f and g are *strongly coprime* if $f(t^k)$ and $g(t^l)$ are relatively prime for all positive integers k and l . We provide some sufficient conditions for strong irreducibility and prove that the Alexander polynomials of twist knots are pairwise strongly coprime and that most of them are strongly irreducible. We apply these results to describe the structure of the subgroup of the rational knot concordance group generated by the twist knots and to provide an explicit set of knots which represent linearly independent elements deep in the solvable filtration of the knot concordance group.

1. INTRODUCTION

A knot is an oriented locally flat embedding of S^1 into S^3 . Modulo slice knots, i.e. knots which bound a locally flat embedding of the 2-disk in the 4-ball, the set of knots forms a group. This group is called the *knot concordance group* and is denoted \mathcal{C} . In [Lev69], Levine defines a surjection from \mathcal{C} to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$. Knots in the kernel of this map are called algebraically slice. The quotient of the knot concordance group by algebraically slice knots is called the algebraic concordance group.

Levine's work also shows that the Alexander polynomial of a knot, $\Delta_K(t) \in \mathbb{Z}[t]$, fits very well into the theory of algebraic concordance. If a knot has irreducible Alexander polynomial then it is not algebraically slice. Moreover, if a pair of knots J and K have coprime Alexander polynomials, then $J\#K$ is algebraically slice if and only if both of J and K are algebraically slice.

Moreover, in [Cha07], Cha discusses the *rational concordance group* of knots, which we will denote by \mathcal{RC} . A knot is called *rationally slice* if it bounds a disk in a 4-manifold with the rational homology of a ball. The appropriate replacements for irreducibility and coprimality of Alexander polynomials in this setting are stronger analogues: a polynomial $f(t) \in \mathbb{Z}[t]$ is called *strongly irreducible* if $f(t^k)$ is irreducible for all positive integers k . Two polynomials $f(t)$ and $g(t)$ are called *strongly coprime* if for all positive integers k and l , the polynomials $f(t^k)$ and $g(t^l)$ are coprime.

The work of [CHL10], for example Theorem 7.7, shows that if one wishes to distinguish concordance classes of knots in more subtle cases, such as knots that are algebraically slice, then the notions of strong irreducibility and strong coprimality arise. The difficulty of actually performing computations in this setting poses a hurdle in the application of

their work. For example, they are able to produce generating sets for infinite rank free abelian groups which sit deeply in the solvable filtration of the knot concordance group, but cannot prove that any one of these generators does not represent the zero element of the concordance group. In section 5, we apply the algebraic tools of this paper to produce explicit *linearly independent* sets of knots.

In [Cha07], Cha defines an epimorphism from \mathcal{RC} to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$. The quotient of \mathcal{RC} by the kernel of this epimorphism is called the *rational algebraic concordance group*. In section 2, we discuss the manner in which strong coprimality and strong irreducibility fit into this theory. Specifically we prove that if two knots, J and K have strongly coprime Alexander polynomials, then $J\#K$ is rationally algebraically slice if and only if J and K are rationally algebraically slice.

In Section 3 we prove Theorem 3.1, showing that the twist knots, depicted in Figure 1, have pairwise strongly coprime Alexander polynomials. We provide an application to the structure of the group they generate in the rational algebraic concordance group.

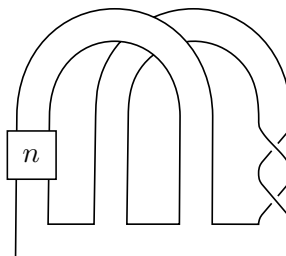


FIGURE 1. The n -twist knot, T_n .

Theorem (Theorem 3.1). *For all positive integers $m \neq n$, the Alexander polynomials Δ_{T_n} and Δ_{T_m} are strongly coprime.*

It is known that Δ_{T_n} is reducible precisely when $n = y(y+1)$ with $y \in \mathbb{Z}$; in fact, n is of this form if and only if T_n is algebraically slice (see [CG86]). We go on in Section 4 to prove that most of the twist knots have strongly irreducible Alexander polynomials.

Theorem (Corollary 4.5). *For every positive integer n that is not a perfect power and not of the form $y(y+1)$ with $y \in \mathbb{Z}$, the Alexander polynomial Δ_{T_n} is strongly irreducible. When n is a perfect square, Δ_{T_n} is not strongly irreducible.*

In order to prove this theorem we develop the following sufficient conditions for strong irreducibility.

Theorem (Corollary 4.2). *Let $f = c_d t^d + \cdots + c_0 \in \mathbb{Z}[t]$ be an irreducible polynomial of degree d , where the coefficients do not all share a common factor. Then if some prime p divides c_d or c_0 exactly once, f is strongly irreducible.*

Theorem (Corollary 4.4). *Let $f(t) = c_d t^d + \cdots + c_1 t + c_0 \in \mathbb{Z}[x]$, where c_0 and c_1 are relatively prime non-zero integers. If f is irreducible and $c_0 \neq \pm \alpha^k$ for any integer α and natural number $k > 1$, then f is strongly irreducible.*

2. APPLICATIONS TO RATIONAL KNOT CONCORDANCE

In [Cha07], Jae Choon Cha defines the rational algebraic concordance group of knots in an analogous manner to the definition of the algebraic concordance group in [Lev69]. Cha defines a complete set of invariants for the rational algebraic concordance group by taking direct limits of Levine's complete set of invariants of algebraic concordance. In this section we prove that these invariants have a splitting property for knots with strongly coprime Alexander polynomials.

An algebraic number z is called *reciprocal* if z and z^{-1} are roots the same irreducible polynomial over \mathbb{Q} . Levine's invariants of algebraic concordance s , e and d are defined as follows:

- For z a reciprocal number with $|z| = 1$, $s_z(K) \in \mathbb{Z}$ is the jump of the Tristram-Levine signature function at z .
- For z a reciprocal number, $e_z(K) \in \mathbb{Z}/2\mathbb{Z}$ is the number of times that the irreducible polynomial of z divides the Alexander polynomial of K , reduced mod 2.
- For z a reciprocal number, $d_z(K) \in \frac{\mathbb{Q}(z+z^{-1})^\times}{\{u\bar{u} \mid u \in \mathbb{Q}(z)^\times\}}$ is the discriminant of the z -primary part of a Seifert matrix of K .

All of these invariants have the property that they vanish when the Alexander polynomial of K is relatively prime to the irreducible polynomial of z [Cha07, Proposition 3.6 (1)]. The invariants s and e are additive under connected sum, while $d_z(K \# J) = (-1)^{e_z(K)e_z(J)} d_z(K) d_z(J)$ [Cha07, Proposition 3.6 (4)].

Let P denote the set of sequences $(a_k)_{k=1}^\infty$ of reciprocal numbers such that $(a_{nk})^n = a_k$ for all n, k . Let P_0 be the subset of P given by adding the restriction that $|a_k| = 1$ for all k . For $a = (a_k) \in P_0$ and $b = (b_k) \in P$, Cha defines $s_a(K) = (s_{a_k}(K))_{k=1}^\infty$, $e_b(K) = (e_{b_k}(K))_{k=1}^\infty$ and $d_b(K) = (d_{b_k}(K))_{k=1}^\infty$. These form a complete set of invariants for rational algebraic concordance [Cha07, Theorem 3.13].

Two polynomials $f(t), g(t) \in \mathbb{Z}[t]$ are called *strongly coprime* if for every pair of integers k and l , the polynomials $f(t^k)$ and $g(t^l)$ have no common roots. We now prove a connection between rational algebraic concordance and the condition of strong coprimality.

Proposition 2.1. *If two knots J and K have strongly coprime Alexander polynomials, then $J \# K$ is rationally algebraically slice if and only if both of J and K are.*

Proof. It follows immediately from the additivity (up to sign) of s , e and d that if both J and K are rationally algebraically slice, then so is $J \# K$. Assume now that K is not rationally algebraically slice. Then there exists some $a = (a_k) \in P$ such that one of $s_a(K)$, $e_a(K)$ or $d_a(K)$ is nonzero, so for some term a_n in the sequence a , one of $s_{a_n}(K)$, $e_{a_n}(K)$ or $d_{a_n}(K)$ is nonzero. This implies that a_n is a root of $\Delta_K(t)$, the Alexander polynomial of K . If one of $s_a(J)$, $e_a(J)$ or $d_a(J)$ is nonzero then similarly a_m is a root of $\Delta_J(t)$ for some m . Then $(a_{mn})^m = a_n$ so that a_{mn} is a root of $\Delta_K(t^m)$. Similarly, a_{mn} is a root of $\Delta_J(t^n)$, contradicting the assumption that Δ_K and Δ_J are strongly coprime. Thus, it must be that $s_{a_n}(J)$, $e_{a_n}(J)$ and $d_{a_n}(J)$ vanish.

By additivity, we have $s_a(J\#K) = s_a(K)$, $e_a(J\#K) = e_a(K)$, and $d_a(J\#K) = d_a(K)$. By assumption, one of these is nonzero, so we conclude that $J\#K$ is not rationally algebraically slice. \square

Corollary 2.2. *If knots K_1, K_2, \dots have strongly irreducible Alexander polynomials that are distinct up to substitutions $f(t) \mapsto \pm f(t^k)$, then the map from the algebraic concordance group to the rational algebraic concordance group is injective on their span.*

Proof. Since distinct strongly irreducible polynomials are strongly coprime, Proposition 2.1 implies that the only way a linear combination $\sum_{j=1}^n c_j K_j$ with $c_j \in \mathbb{Z}$ can be rationally algebraically slice is if $c_j K_j$ is rationally algebraically slice for all j .

Now, we note that K_j has infinite order in the algebraic concordance group if and only if $s_z(K_j)$ is nonzero for some reciprocal number z with $|z| = 1$. By [Cha07, Proposition 6.1 (1)], z must be a root of $\Delta_{K_j}(t)$. Let $a = (a_n) = (\sqrt[n]{z})$ be any compatible sequence of n th roots of z . Then each a_n is reciprocal, since a_n and a_n^{-1} are both roots of the same irreducible polynomial $\Delta_{K_j}(t^n)$. Thus, $a \in P_0$ and certainly $s_a(K_j)$ is nonzero, since its first entry is nonzero. Since s is a homomorphism to a torsion-free group it follows that K_j is of infinite order in the rational algebraic concordance group.

The proof in the case that K_j is of order two or four is the same, but uses the invariants e and d in place of s . \square

3. STRONG COPRIMALITY

Recall that the Alexander polynomial of the n -twist knot T_n is

$$\Delta_{T_n}(t) = nt^2 - (2n+1)t + n.$$

In this section, we prove the following theorem:

Theorem 3.1. *For positive integers $m \neq n$, the Alexander polynomials Δ_{T_n} and Δ_{T_m} are strongly coprime.*

Corollary 3.2. *If some linear combination $\sum_{j=1}^m c_j T_j$ is rationally algebraically slice, then each $c_j T_j$ is rationally algebraically slice.*

Proof. This follows immediately from Theorem 3.1 and Proposition 2.1. \square

Proof of Theorem 3.1. By the quadratic formula, the roots of $\Delta_{T_n}(t)$ are given by r_n and $1/r_n$, where $r_n = \frac{2n+1+\sqrt{4n+1}}{2n}$.

If these polynomials were not strongly coprime, then for nonzero integers k, l ,

$$(1) \quad r_n^k = r_m^l.$$

Since $r_n > 1$, it must be that $\text{sign}(k) = \text{sign}(l)$, from here on we assume both are positive. If k and l had a common factor d , we could take the positive real d th root of both sides of this equation. We thus may assume that k and l are relatively prime. The proof now proceeds by cases.

Lemma 3.3. *If $(4n+1)(4m+1)$ is not the square of an integer, then Δ_{T_n} and Δ_{T_m} are strongly coprime.*

Proof. Otherwise, (1) holds so that $r_n^k = r_m^l$. An easy inductive argument shows that for each $k > 0$ there are positive rationals a and b such that $\left(\frac{2n+1+\sqrt{4n+1}}{2n}\right)^k = a + b\sqrt{4n+1}$. Thus, there are positive rational numbers a, b, c, d such that

$$a + b\sqrt{4n+1} = c + d\sqrt{4m+1},$$

and rearranging this equation we see that

$$b\sqrt{4n+1} - d\sqrt{4m+1} = c - a.$$

By squaring both sides of this equation and performing arithmetic we get

$$b^2(4n+1) + d^2(4m+1) - 2bd\sqrt{(4n+1)(4m+1)} = (c-a)^2$$

so that since bd is a nonzero rational number, $\sqrt{(4n+1)(4m+1)}$ is rational, contradicting the assumption that $(4n+1)(4m+1)$ is not a square. \square

Thus, we are reduced to the case that $(4n+1)(4m+1)$ is a square. This in particular implies that there are odd integers a, b, D , with D squarefree and congruent to 1 mod 4 such that $4n+1 = a^2D$ and $4m+1 = b^2D$. Making these substitutions,

$$(2) \quad r_n = \frac{a^2D - 1 + 2(1 + a\sqrt{D})}{(a\sqrt{D} + 1)(a\sqrt{D} - 1)} = \frac{a\sqrt{D} + 1}{a\sqrt{D} - 1}, \quad r_m = \frac{b\sqrt{D} + 1}{b\sqrt{D} - 1}.$$

We first deal with the case where $D = 1$, i.e. where r_n and r_m are both rational numbers, namely

$$r_n = \frac{a+1}{a-1} = \frac{2y+2}{2y} = \frac{y+1}{y},$$

where $a = 2y+1$, and $r_m = \frac{z+1}{z}$ with $b = 2z+1$. We thus have

$$r_n^k = r_{y(y+1)}^k = \left(\frac{y+1}{y}\right)^k = \left(\frac{z+1}{z}\right)^l = r_{z(z+1)}^l = r_m^l,$$

and comparing prime factorizations, we see that we must have $y = \alpha^l$ and $z = \alpha^k$ for some natural number $\alpha > 1$. But then if $k < l$ we have

$$\left(\frac{y+1}{y}\right)^k = \left(1 + \frac{1}{\alpha^l}\right)^k < \left(1 + \frac{1}{\alpha^k}\right)^k < \left(1 + \frac{1}{\alpha^k}\right)^l = \left(\frac{z+1}{z}\right)^l,$$

a contradiction.

The proof in the case $D > 1$ works exactly the same way, but since r_n and r_m are elements of $\mathbb{Q}(\sqrt{D})$ and instead of \mathbb{Q} , we must replace unique prime factorization in \mathbb{Z} with unique factorization into prime ideals (see [FT93] II.1, p. 54) in the ring $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ of integers of the number field $\mathbb{Q}(\sqrt{D})$.

To this end, let p be a prime number dividing n . Then the ideal generated by p in $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ is not prime. It factors as $\langle p \rangle = P_1 P_{-1}$, where

$$P_1 = \left\langle p, \frac{a\sqrt{D}+1}{2} \right\rangle, \quad P_{-1} = \left\langle p, \frac{a\sqrt{D}-1}{2} \right\rangle$$

In order to see this, we expand $P_1 P_{-1} = \left\langle p^2, p\frac{a\sqrt{D}+1}{2}, p\frac{a\sqrt{D}-1}{2}, \frac{a^2 D-1}{4} \right\rangle$. The first, second and third of these generators are clearly in $\langle p \rangle$. The fourth, $\frac{a^2 D-1}{4} = n$ is a multiple of p by design. Thus, $\langle p \rangle \supseteq P_1 P_{-1}$. In order to see the opposite containment, note that the difference between the second and third of the generators is exactly p .

Recall that for an ideal I in $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$, the norm $N(I)$ is defined to be the order of the quotient $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right] / I$ as an abelian group. We will require the following basic properties of the norm (see [FT93] pp. 56-57): for ideals $I, J \subseteq \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$,

- if $I = \langle r + s\sqrt{D} \rangle$ is principal, then $N(I) = |N(r + s\sqrt{D})| = |r^2 - Ds^2|$,
- if $I \supsetneq J$, then $N(I)$ is a proper divisor of $N(J)$,
- if $N(I)$ is a prime integer, then I is a prime ideal, and
- the norm is multiplicative: $N(IJ) = N(I)N(J)$.

Thus, since

$$N(P_1)N(P_{-1}) = N(\langle p \rangle) = p^2$$

with $N(P_1) < p^2$ and $N(P_{-1}) < p^2$, we must have $N(P_1) = N(P_{-1}) = p$ and both P_1 and P_{-1} are prime ideals.

Lemma 3.4. *If p^k divides n then for $\epsilon = \pm 1$,*

$$P_\epsilon^k = \left\langle p^k, \frac{a\sqrt{D}+\epsilon}{2} \right\rangle.$$

Proof. We proceed by induction; the lemma holds for $k = 1$ by the definition of P_ϵ . If it holds for $k - 1$, then

$$\begin{aligned} P_\epsilon^k &= P_\epsilon^{k-1} P_\epsilon \\ &= \left\langle p^{k-1}, \frac{a\sqrt{D}+\epsilon}{2} \right\rangle \left\langle p, \frac{a\sqrt{D}+\epsilon}{2} \right\rangle \\ &= \left\langle p^k, p\frac{a\sqrt{D}+\epsilon}{2}, n + \epsilon\frac{a\sqrt{D}+\epsilon}{2} \right\rangle, \end{aligned}$$

and since p^k divides n , we have

$$\begin{aligned} P_\epsilon^k &= \left\langle p^k, p\frac{a\sqrt{D}+\epsilon}{2}, \frac{a\sqrt{D}+\epsilon}{2} \right\rangle \\ &= \left\langle p^k, \frac{a\sqrt{D}+\epsilon}{2} \right\rangle. \end{aligned}$$

completing the proof. □

Returning to the situation of interest, we determine the multiplicity with which P_1 divides the numerators and denominators of (2) in terms of the multiplicity with which p divides m and n .

Lemma 3.5. *Assume that p divides n with multiplicity $k > 0$ over the integers, then $\frac{a\sqrt{D}+1}{2}$ is contained in P_1^k but not P_1^{k+1} and $\frac{a\sqrt{D}-1}{2}$ is not contained in P_1 .*

Proof. First, note that if $\frac{a\sqrt{D}-1}{2} \in P_1$ then since $\frac{a\sqrt{D}+1}{2} \in P_1$, it would follow that $1 \in P_1$, contradicting that P_1 is a proper ideal.

By Lemma 3.4, $\frac{a\sqrt{D}+1}{2} \in P_1^k$. In order to show that no greater power of P_1 contains $\frac{a\sqrt{D}+1}{2}$, we perform a norm computation:

$$N\left(\left\langle \frac{a\sqrt{D}+1}{2} \right\rangle\right) = \left| \frac{(1+a\sqrt{D})(1-a\sqrt{D})}{4} \right| = n.$$

Recalling that $N(P_1) = p$, we note that if $\frac{a\sqrt{D}+1}{2}$ were contained in P_1^{k+1} then it would follow that p^{k+1} divides n , in contradiction to the assumption to the contrary. This completes the proof. \square

Now suppose that (1) holds, so that in light of (2):

$$(3) \quad \left(\frac{a\sqrt{D}+1}{2} \middle/ \frac{a\sqrt{D}-1}{2} \right)^k = \left(\frac{b\sqrt{D}+1}{2} \middle/ \frac{b\sqrt{D}-1}{2} \right)^l$$

Cross multiplying reduces this to

$$(4) \quad \left(\frac{a\sqrt{D}+1}{2} \right)^k \left(\frac{b\sqrt{D}-1}{2} \right)^l = \left(\frac{a\sqrt{D}-1}{2} \right)^k \left(\frac{b\sqrt{D}+1}{2} \right)^l.$$

First, suppose that p is a prime factor of n but is not a factor of m . By Lemma 3.5, $\frac{a\sqrt{D}-1}{2} \notin P_1$. Since the norm of $\frac{b\sqrt{D}-1}{2}$ is m which is not divisible by $p = N(P_1)$, we must have $\frac{b\sqrt{D}-1}{2} \notin P_1$. Thus, the right-hand side of (4) is not an element of the prime ideal P_1 , but we know the left-hand side is because $\frac{a\sqrt{D}+1}{2} \in P_1$, a contradiction. Thus, n and m must have the same set of prime factors.

Suppose then that p is a prime factor of both n and m . Let $x = v_p(n)$ be the highest power such that p^x divides n and likewise let $y = v_p(m)$ be the highest power of p such that p divides m . We have two factorizations of the ideal $\langle p \rangle$ in the Dedekind domain $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$:

$$\begin{aligned} \langle p \rangle &= P_1 P_{-1} \quad \text{where} \quad P_1 = \left\langle p, \frac{a\sqrt{D}+1}{2} \right\rangle, \quad P_{-1} = \left\langle p, \frac{a\sqrt{D}-1}{2} \right\rangle, \quad \text{and} \\ \langle p \rangle &= \tilde{P}_1 \tilde{P}_{-1} \quad \text{where} \quad \tilde{P}_1 = \left\langle p, \frac{b\sqrt{D}+1}{2} \right\rangle, \quad \tilde{P}_{-1} = \left\langle p, \frac{b\sqrt{D}-1}{2} \right\rangle. \end{aligned}$$

By unique factorization of ideals, either $P_1 = \tilde{P}_1$ or $P_1 = \tilde{P}_{-1}$. In the latter case, by Lemma 3.5, P_1 divides the left hand side of (4) with multiplicity $xk + yl$ and the right hand side with multiplicity 0, which is a contradiction. Thus, $P_1 = \tilde{P}_1$ and P_1 divides the

left hand side with multiplicity xk and the left hand side with multiplicity yl . It follows that $xk = yl$.

Since we can assume that k and l are relatively prime, $xk = yl$ implies that there is some integer c (depending on p) with $y = ck$ and $x = cl$. Thus, n and m have related prime factorizations:

$$(5) \quad \begin{aligned} n &= p_1^{c_1 l} p_2^{c_2 l} \dots p_f^{c_f l} = (p_1^{c_1} p_2^{c_2} \dots p_f^{c_f})^l = \alpha^l, \text{ and} \\ m &= p_1^{c_1 k} p_2^{c_2 k} \dots p_f^{c_f k} = (p_1^{c_1} p_2^{c_2} \dots p_f^{c_f})^k = \alpha^k, \end{aligned}$$

for some integer $\alpha = p_1^{c_1} p_2^{c_2} \dots p_f^{c_f} > 1$.

Now, having analyzed all the finite primes, we consider the “infinite prime”, i.e. we show using an inequality that $(r_n)^k = (r_m)^l$ is impossible over the real numbers for $n = \alpha^l$ and $m = \alpha^k$. Let α be any integer greater than 1. Exchanging n and m if necessary, we may assume $k < l$. Substituting $n = \alpha^l$ and $m = \alpha^k$ in to the original expressions for r_n and r_m from the quadratic formula, we get

$$\left(\frac{2\alpha^l + 1 + \sqrt{4\alpha^l + 1}}{2\alpha^l} \right)^k = \left(\frac{2\alpha^k + 1 + \sqrt{4\alpha^k + 1}}{2\alpha^k} \right)^l.$$

Making a simplification, this is equivalent to

$$\left(1 + \frac{1}{2\alpha^l} + \sqrt{\frac{1}{\alpha^l} + \frac{1}{4\alpha^{2l}}} \right)^k = \left(1 + \frac{1}{2\alpha^k} + \sqrt{\frac{1}{\alpha^k} + \frac{1}{4\alpha^{2k}}} \right)^l.$$

Observe that for $l > 0$, the real number $1 + \frac{1}{2\alpha^l} + \sqrt{\frac{1}{\alpha^l} + \frac{1}{4\alpha^{2l}}}$ is greater than 1 and that this number increases as l decreases. Thus, since $k < l$,

$$\left(1 + \frac{1}{2\alpha^l} + \sqrt{\frac{1}{\alpha^l} + \frac{1}{4\alpha^{2l}}} \right)^k < \left(1 + \frac{1}{2\alpha^k} + \sqrt{\frac{1}{\alpha^k} + \frac{1}{4\alpha^{2k}}} \right)^k < \left(1 + \frac{1}{2\alpha^k} + \sqrt{\frac{1}{\alpha^k} + \frac{1}{4\alpha^{2k}}} \right)^l.$$

In particular, this implies that these are not equal and completes the proof of Theorem 3.1. \square

4. STRONG IRREDUCIBILITY

A polynomial $f \in \mathbb{Q}[t]$ is *strongly irreducible* if $f(t^k)$ is irreducible for all positive integers k . In this section, we apply the idea behind Eisenstein’s Irreducibility Criterion to prove various criteria that guarantee strong irreducibility. Recall ([Gou97], 2.1) that for $p \in \mathbb{Z}$ prime, the p -adic valuation on \mathbb{Q} is the function $v_p: \mathbb{Q} \rightarrow \mathbb{Z} \cup \{+\infty\}$ defined by

$$v_p(x) = \begin{cases} k & x = \frac{r}{s} \cdot p^k \text{ where } k, r, s \in \mathbb{Z} \text{ satisfy } p \nmid r, p \nmid s, \\ +\infty & x = 0. \end{cases}$$

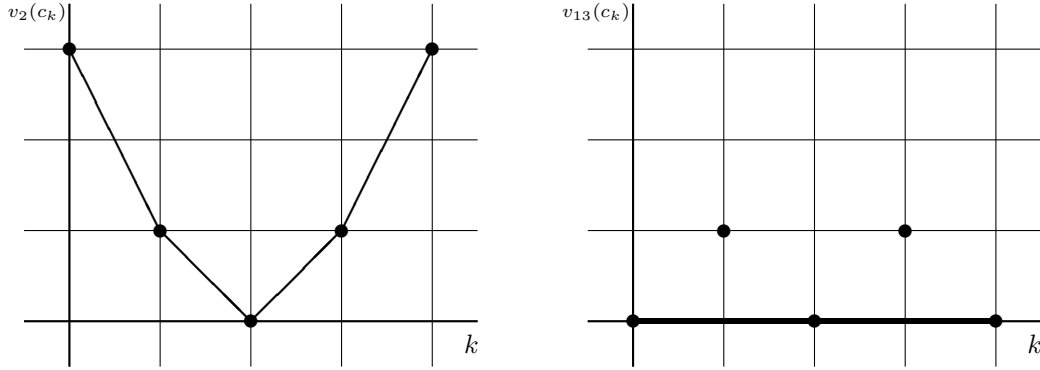
Theorem 4.1 (An Eisenstein criterion for strong irreducibility). *Let $f = c_d t^d + \dots + c_0$ be an irreducible polynomial in $\mathbb{Q}[t]$. Let p be a prime number and suppose that for some*

$0 \leq i < j \leq d$, the following conditions are satisfied, where $a = j - i$, $b = v_p(c_j) - v_p(c_i)$, and $m = b/a$:

- (1) c_i and c_j are non-zero,
- (2) $v_p(c_i) \neq v_p(c_j)$,
- (3) a and b are relatively prime,
- (4) $p \nmid b$, and
- (5) $v_p(c_k) \geq m(k - i) + v_p(c_i)$ for $i < k < j$, and
 $v_p(c_k) > m(k - i) + v_p(c_i)$ for $0 \leq k < i$ or $j < k \leq d$.

Then if $f(t^{|b|})$ is irreducible, f is strongly irreducible.

We defer the proof of Theorem 4.1 until the end of the section. To make sense of the hypotheses, and especially condition (5), we recall the definition of the *Newton polygon of f at the prime p* (see [Gou97], 6.4): we plot the points $(k, v_p(c_k))$ in the plane for $0 \leq k \leq d$, and take the lower part of the boundary of their convex hull.¹ For example, if $f(t) = 8t^4 - 26t^3 + 35t^2 - 26t + 8$ is the Alexander polynomial of the knot $12a_{1163}$ from [CL], then the Newton polygon of f at the primes $p = 2$ and $p = 13$ are the following:

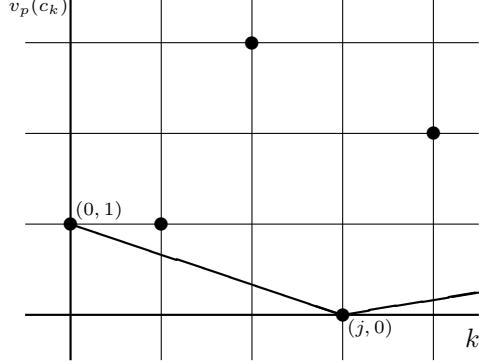


In terms of the Newton polygon for f at p , we can restate conditions (1)-(5) as saying that the Newton polygon has some edge which (2) is not horizontal, (3) does not pass through a lattice point besides its endpoints, and (4) has a vertical height b that is not divisible by p . In the case of the above example, the edge $((1, 1), (2, 0))$ of the Newton polygon at $p = 2$ satisfies conditions (1)-(5) with $b = 1$. Thus, since f is irreducible, by Theorem 4.1, f is strongly irreducible. Since $12a_{1163}$ is of infinite order in the algebraic concordance group, Corollary 2.2 implies that it is of infinite order in the rational algebraic concordance group.

Corollary 4.2. *Let $f = c_d t^d + \cdots + c_0 \in \mathbb{Z}[t]$ be an irreducible polynomial of degree d , where we assume the coefficients do not all share a common factor. Then if some prime p divides c_d or c_0 exactly once, f is strongly irreducible.*

¹We omit the points $(k, +\infty)$ arising from coefficients $c_k = 0$, since these points would not affect the lower boundary of the convex hull.

Proof. Suppose that p divides c_0 exactly once and that j is the first index for which c_j is not divisible by p .



Then $((0, 1), (j, 0))$ is an edge of the Newton polygon for f at p , with $a = j$, $b = -1$, so certainly a and b are relatively prime, $p \nmid b$, and the hypotheses of Theorem 4.1 are satisfied. We conclude f is strongly irreducible.

Similarly, in the case that instead p divides c_d exactly once, we consider the edge $((i, 0), (d, 1))$, where i is the last index for which c_i is not divisible by p . \square

Note that the previous strong irreducibility result of Jae Choon Cha follows immediately from Corollary 4.2:

Proposition ([Cha07], Proposition 3.18). *Suppose $\lambda(t) = pt^2 - (2p + k)t + p$, where p is a prime and k is an integer such that $k \not\equiv 0 \pmod{p}$ and $k \not\equiv -2p \pm 1 \pmod{p^2}$. Then $\lambda(t)$ is strongly irreducible.*

We need the second congruence condition only to guarantee that $\pm \frac{1}{p}$, $\pm p$ are not the roots of $\lambda(t)$, and that therefore $\lambda(t)$ is irreducible over \mathbb{Q} . Indeed, this means that the hypothesis $k \not\equiv -2p \pm 1 \pmod{p^2}$ in the proposition can be weakened to $k \not\equiv -2p \pm (1 + p^2)$.

We now state a second strong irreducibility criterion that takes into account data at several primes.

Theorem 4.3. *Let $f = c_d t^d + \cdots + c_0 \in \mathbb{Q}[x]$ be irreducible. Assume that for every prime q there exists a prime p and an edge $((i, v_p(c_i)), (i + a, v_p(c_i) + b))$ of the Newton polygon for f at p such that the numbers a and b are relatively prime and $q \nmid b$. Then f is strongly irreducible.*

Again, we defer the proof until the end of the section.

Corollary 4.4. *Let $f(t) = c_d t^d + \cdots + c_1 t + c_0 \in \mathbb{Z}[x]$, where c_0 and c_1 are relatively prime non-zero integers. If f is irreducible and $c_0 \neq \pm \alpha^k$ for any integer α and natural number $k > 1$, then f is strongly irreducible.*

Proof. Let $c_0 = \pm p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ be the prime factorization of n . Then for any prime q , since c_0 is not a q th power up to sign, there is some i for which $q \nmid e_i$. At the prime $p = p_i$, the Newton polygon for f has an edge $((0, e_i), (1, 0))$, for which $a = 1$ so a and b are certainly relatively prime, and $b = -e_i$ is not divisible by q . Thus, the hypotheses of Theorem 4.3 are satisfied and f is strongly irreducible. \square

As in Corollary 4.2, we could replace c_0 and c_1 with c_d and c_{d-1} in the statement of Corollary 4.4, either by giving the analogous proof or by replacing f with $t^d f(t^{-1})$.

Corollary 4.5. *If a natural number n is not a perfect power and $n \neq y(y+1)$ for any $y \in \mathbb{Z}$, then the Alexander polynomial*

$$\Delta_{T_n}(t) = nt^2 - (2n+1)t + n$$

is strongly irreducible.

Proof. This follows immediately from Corollary 4.4: the $n \neq y(y+1)$ condition is equivalent to the irreducibility of $\Delta_{T_n}(t)$ over \mathbb{Q} (see the analysis of the $D = 1$ case in the proof of Theorem 3.1). \square

Remark 4.6. In the case where $n = \alpha^2$ is a perfect square, the polynomial $\Delta_{T_n}(t)$ fails to be strongly irreducible:

$$\Delta_{T_n}(t^2) = \alpha^2 t^4 - (2\alpha^2 + 1)t^2 + \alpha^2 = (\alpha t^2 - t - \alpha)(\alpha t^2 + t - \alpha).$$

In fact, in this case, the n -twist knot is rationally algebraically slice. In order to see this, we check that Cha's three invariants s , e , and d all vanish. If s were nonzero then for some $a = (a_j) \in P_0$, the invariant $s_{(a)}(T_n)$ would not be identically zero. Thus, $s_{a_j}(T_n)$ would be nonzero for some a_j , implying that a_j is a root of Δ_{T_n} . Since $a \in P_0$, the term a_{2j} is a square root of a_j , and hence must satisfy one of the irreducible factors of $\Delta_{T_n}(t^2)$. But neither of these factors are symmetric, which implies that a_{2j} is not reciprocal, contradicting that (a) is in P_0 . The argument that d and e vanish is the same.

On the other hand, when n is a perfect power but not a perfect square, we have found no examples where $\Delta_{T_n}(t)$ fails to be strongly irreducible, but it seems the techniques of this section can't easily be applied in this case in general.

For example, when $n = 6^3 = 216$, the polynomial $\Delta_{T_{216}}(t^3)$ has roots in both \mathbb{Q}_2 and \mathbb{Q}_3 , i.e. in \mathbb{Q}_p for every p at which the Newton polygon is not simply a single horizontal segment. On the other hand, $\Delta_{T_{216}}(t)$ is nevertheless strongly irreducible: applying Theorem 4.1 to $\Delta_{T_{216}}(t)$ at $p = 2$, we see that we need only check that $\Delta_{T_{216}}(t^3)$ is irreducible over \mathbb{Q} , which is easily checked by computer (in fact, it's irreducible mod 11).

Applying Corollary 2.2, the above discussion yields the following:

Corollary 4.7. *If K is the kernel of the map from the subgroup generated by $\{T_n \mid n > 0\}$ in the algebraic concordance group to the rational algebraic concordance group, then*

$$\langle \{T_n \mid n \text{ is a perfect square}\} \rangle \subseteq K \subseteq \langle \{T_n \mid n \text{ is a perfect power}\} \rangle.$$

Numerical evidence suggests that $K = \langle \{T_n \mid n \text{ is a perfect square}\} \rangle$.

Proof of Theorem 4.1. Let $\mathbb{Q}(r)$ be the extension of \mathbb{Q} obtained by adjoining a root r of the irreducible polynomial f . Then $[\mathbb{Q}(r) : \mathbb{Q}] = d$, and to show that the degree xd polynomial $f(t^x)$ is irreducible over \mathbb{Q} , it suffices to show that $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$ for some x th root $\sqrt[x]{r}$ of r . If $x = yz$, then

$$[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = [\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(\sqrt[y]{r})][\mathbb{Q}(\sqrt[y]{r}) : \mathbb{Q}(r)],$$

where $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(\sqrt[y]{r})] \leq z$ and $[\mathbb{Q}(\sqrt[y]{r}) : \mathbb{Q}(r)] \leq y$. Therefore, if $f(t^x)$ is irreducible, or equivalently $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$, then $[\mathbb{Q}(\sqrt[y]{r}) : \mathbb{Q}(r)] = y$ and $f(t^y)$ is irreducible as well.

Thus, to show f is strongly irreducible, it suffices to show that $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$ when $x = |b|z$ is a multiple of b , and since we are assuming that $f(t^{|b|})$ is irreducible, it suffices to show that $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(\sqrt[|b|]{r})] = z$.

Now, applying standard results on Newton polygons (see [Gou97] 6.4, especially Theorem 6.4.7), we see that over the p -adics, f has a factor $g \in \mathbb{Q}_p[t]$ of degree a , such that every root $\tilde{r} \in \overline{\mathbb{Q}_p}$ of g satisfies $v_p(\tilde{r}) = m = b/a$, where the valuation $v_p : \overline{\mathbb{Q}_p} \rightarrow \mathbb{Q} \cup \{+\infty\}$ is the extension of $v_p : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{+\infty\}$. Since f is irreducible over \mathbb{Q} and $\overline{\mathbb{Q}_p}$ is algebraically closed, there exists an inclusion $\mathbb{Q}(r) \hookrightarrow \overline{\mathbb{Q}_p}$ taking r to \tilde{r} . Thus, we identify $\mathbb{Q}(r)$ with a subfield of $\overline{\mathbb{Q}_p}$, where r is a root of g .

Now, since a and b are relatively prime and $v_p(r) = b/a$, the ramification index $e(\mathbb{Q}_p(r)/\mathbb{Q}_p)$ is a multiple of a , but r is a root of a polynomial g of degree a . Therefore g is irreducible and $e(\mathbb{Q}_p(r)/\mathbb{Q}_p) = a$. (For basic facts on ramified and unramified extensions and ramification index, see [Gou97] 5.4.)

Let π be a uniformizer of $\mathbb{Q}_p(r)$. Then $\mathbb{Q}_p(\sqrt[|b|]{r}) = \mathbb{Q}_p\left(r, \sqrt[|b|]{r\pi^{-b}}\right)$, so since $v_p(r\pi^{-b}) = 0$ and $p \nmid b$, the extension $\mathbb{Q}_p(\sqrt[|b|]{r})/\mathbb{Q}_p(r)$ is unramified. On the other hand, $v_p(\sqrt[x]{r}) = \frac{m}{x} = \frac{b}{az|b|} = \pm \frac{1}{az}$, so $e(\mathbb{Q}_p(\sqrt[x]{r})/\mathbb{Q}_p)$ is a multiple of az . Now, since

$$e(\mathbb{Q}_p(\sqrt[x]{r})/\mathbb{Q}_p) = e(\mathbb{Q}_p(\sqrt[x]{r})/\mathbb{Q}_p(\sqrt[|b|]{r})) \cdot e(\mathbb{Q}_p(\sqrt[|b|]{r})/\mathbb{Q}_p(r)) \cdot e(\mathbb{Q}_p(r)/\mathbb{Q}_p),$$

with $e(\mathbb{Q}_p(\sqrt[|b|]{r})/\mathbb{Q}_p(r)) = 1$ and $e(\mathbb{Q}_p(r)/\mathbb{Q}_p) = a$, we find that $e(\mathbb{Q}_p(\sqrt[x]{r})/\mathbb{Q}_p(\sqrt[|b|]{r}))$ is a multiple of z . Therefore $[\mathbb{Q}_p(\sqrt[x]{r}) : \mathbb{Q}_p(\sqrt[|b|]{r})]$ is a multiple of z , and hence $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(\sqrt[|b|]{r})]$ is a multiple of z , and since this extension is obtained by taking a z th root, we have $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(\sqrt[|b|]{r})] = z$, as desired. \square

Proof of Theorem 4.3. Let $\mathbb{Q}(r)$ be the extension of \mathbb{Q} obtained by adjoining a root r of the irreducible polynomial f . As in the proof of Theorem 4.1, we must show that for every natural number x , we have $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$ for some x th root $\sqrt[x]{r}$ of r .

We observe that if $x = yz$, then $\mathbb{Q}(\sqrt[x]{r})$ contains subfields $\mathbb{Q}(\sqrt[y]{r})$ and $\mathbb{Q}(\sqrt[z]{r})$. Thus, if $[\mathbb{Q}(\sqrt[y]{r}) : \mathbb{Q}(r)] = y$ and $[\mathbb{Q}(\sqrt[z]{r}) : \mathbb{Q}(r)] = z$, with y and z relatively prime, we could conclude that $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$. It therefore suffices to prove that $[\mathbb{Q}(\sqrt[x]{r}) : \mathbb{Q}(r)] = x$ in the case where $x = q^h$ is a prime power.

Now, let p be the prime associated to q in the hypotheses of the theorem. As in the proof of Theorem 4.1, we may conclude that over \mathbb{Q}_p , the polynomial f has a factor g whose roots $\tilde{r} \in \overline{\mathbb{Q}_p}$ all satisfy $v_p(\tilde{r}) = b/a$. Again, since f is irreducible over \mathbb{Q} we can identify

$\mathbb{Q}(r)$ with a subfield of $\overline{\mathbb{Q}_p}$ in which r is a root of g . Again, since a and b are relatively prime, g is irreducible over \mathbb{Q}_p and $\mathbb{Q}_p(r)/\mathbb{Q}_p$ is totally ramified of degree a .

Now, since $v_p(\sqrt[aq^h]{r}) = \frac{b}{aq^h}$ and $q \nmid b$, the ramification index $e\left(\mathbb{Q}_p(\sqrt[aq^h]{r})/\mathbb{Q}_p\right)$ is a multiple of aq^h . Therefore, since

$$e\left(\mathbb{Q}_p(\sqrt[aq^h]{r})/\mathbb{Q}_p\right) = e\left(\mathbb{Q}_p(\sqrt[aq^h]{r})/\mathbb{Q}_p(r)\right) \cdot e\left(\mathbb{Q}_p(r)/\mathbb{Q}_p\right),$$

and $e(\mathbb{Q}_p(r)/\mathbb{Q}_p) = a$, we conclude that $e\left(\mathbb{Q}_p(\sqrt[aq^h]{r})/\mathbb{Q}_p(r)\right)$ is a multiple of q^h and hence $[\mathbb{Q}_p(\sqrt[aq^h]{r}) : \mathbb{Q}_p(r)]$ and $[\mathbb{Q}(\sqrt[aq^h]{r}) : \mathbb{Q}(r)]$ are multiples of q^h . Since $[\mathbb{Q}(\sqrt[aq^h]{r}) : \mathbb{Q}(r)] \leq q^h$, we conclude $[\mathbb{Q}(\sqrt[aq^h]{r}) : \mathbb{Q}(r)] = q^h$ as desired. \square

5. APPLICATION: MAKING THE [CHL10] CONSTRUCTION OF KNOTS DEEP IN THE SOLVABLE FILTRATION EXPLICIT.

In [COT03], Cochran, Orr and Teichner define a filtration of the knot concordance group indexed by half-integers:

$$\cdots \subseteq \mathcal{F}_{1.5} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}.$$

In [CHL10], Cochran, Harvey and Leidy perform an iterated infection process along a particular class of ribbon knots to yield subgroups of $\mathcal{F}_n/\mathcal{F}_{n.5}$ isomorphic to \mathbb{Z}^∞ . We recall their construction:

Definition (Definition 7.2 of [CHL10]). *Given a ribbon knot R in S^3 and an unknotted curve η in $S^3 - R$ such that the linking number $\text{lnk}(R, \eta) = 0$, the pair (R, η) , also written R_η , is called a robust doubling operator if:*

- (1) *the rational Alexander module of R , $A_0(R)$ is cyclic, generated by η , and $A_0(R) \cong \frac{\mathbb{Q}[t, t^{-1}]}{\delta(t)\delta(t^{-1})}$ for a prime polynomial δ , and*
- (2) *for each isotropic submodule P of $A_0(R)$, with the first order signature corresponding to P , $\rho(R, \phi_P)$, vanishes or P corresponds to a ribbon disk for R .*

The justification for calling the pair (R, η) an operator is a procedure called *infection* which takes a doubling operator R_η and a knot J and produces a knot $R_\eta(J)$ by cutting the strands of R which pass through a disk bounded by η and tying them into the knot J , as is indicated in Figure 2.

Cochran, Harvey, and Leidy provide examples of robust doubling operators and prove the following theorem:

Theorem (Theorem 7.7, [CHL10]). *Let I be an arbitrary indexing set. Take $\{\mathcal{Q}_i = (q_{i,1}, \dots, q_{i,n})\}_{i \in I}$ to be a collection of n -tuples of polynomials which are termwise strongly coprime, that is, for all $i, j \in I$ and each m , $q_{i,m}$ and $q_{j,m}$ are strongly coprime. Take $\{\mathcal{R}_i = (R_{\alpha_{i,n}}^{i,n} \circ \cdots \circ R_{\alpha_{i,1}}^{i,1})\}_{i \in I}$ to be a set of iterated robust doubling operators such that $\Delta_{R_{i,m}} = q_{i,m}$. For each $i \in I$, take $\mathcal{K}_i = \{K_{i,j}\}_{j=1}^\infty$ to be a sequence of knots with vanishing*



FIGURE 2. Left: A portion of a knot R going through a disk bounded by η . Right: the knot $R_\eta(J)$

Arf-invariants, such that for each $i \in I$, the set of integrals of the Tristram-Levine signature functions $\{\rho_0(K_{i,j})\}_{j=1}^\infty$, is linearly independent of the first order signatures of $R_{i,1}$. Then the set

$$\{\mathcal{R}_i(K_{i,j}) \mid i \in I, 1 \leq j < \infty\}$$

is linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$.

From here on we refer to the integral of the Tristram-Levine signature function of a knot as its ρ_0 -invariant. For a discussion of von Neumann ρ -invariants, see [COT03].

The doubling operators that Cochran, Harvey, and Leidy construct, $R_{\alpha_{i,m}}^{i,m}$, satisfy all of the needed conditions and have only one non-zero first order signature. By picking the sequences \mathcal{K}_i to consist of knots with linearly independent ρ_0 -invariants, the addition of this first order signature introduces at most one relation. Thus, by removing at most one term from the sequence \mathcal{K}_i , the linear independence condition can be met.

Unfortunately, the nonzero first order signatures are difficult to compute, and so it is not yet known which $K_{i,j}$ must be removed from the sequence \mathcal{K}_i to achieve this independence. For a different set of robust doubling operators, we circumvent this difficulty.

Let J_0, J_1, \dots be a sequence of knots with vanishing Arf-invariant and linearly independent ρ_0 -invariant. [COT04, Theorem 2.6] provides such a sequence.

For each positive integer b , consider the amphichiral ribbon knot \tilde{R}^b depicted on the left hand side of Figure 3. It has Alexander module $A_0(\tilde{R}^b) = \frac{\mathbb{Q}[t, t^{-1}]}{\langle \delta_b(t)^2 \rangle}$ where

$$\delta_b(t) = bt^2 - (1 + 2b)t + b.$$

This module has only two isotropic submodules, $P = \langle \delta_b \rangle$ and 0. The corresponding ρ -invariants both vanish: $\rho(\tilde{R}^b, \phi_P) = 0$ since P corresponds to a ribbon disk, and $\rho(\tilde{R}^b, \phi_0) = 0$ by [CHL08, Proposition 4.5] since \tilde{R}^b is amphichiral.

Of course this is problematic: the submodule 0 cannot correspond to a ribbon disk. No choice of η will make \tilde{R}_η^b a robust doubling operator. With that in mind we perform the infection by J_0 depicted on the right hand side of Figure 3. Calling the resulting ribbon knot R^b , we see that $\rho(R^b, \phi_P) = 0$ since R^b is still ribbon and P corresponds to a ribbon disk for R^b . By [COT04, Proposition 3.2], $\rho(R^b, \phi_0) = \rho_0(J_0)$ which in particular

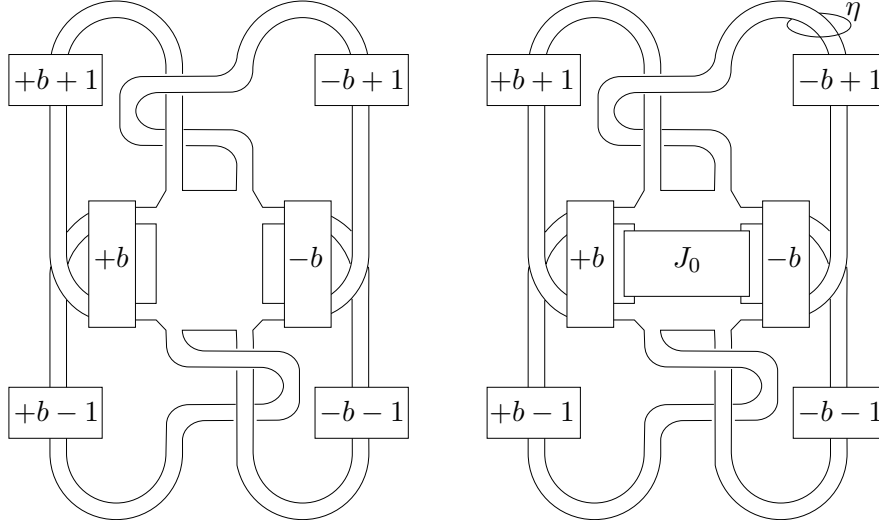


FIGURE 3. On the left: The ribbon knot \tilde{R}^b . On the right: the robust doubling operator R_η^b gotten by infecting \tilde{R}^b by J_0 . The $\pm b$ indicates b positive or negative full twists between bands without adding twists in either band.

is nonzero. Thus, if η is any unknotted curve representing a generator of $A_0(R^b)$ (such as the one depicted in Figure 3), R_η^b is a robust doubling operator

Notice that since $\delta_b = \Delta_{T_b}$ is exactly the Alexander polynomial of the b -twist knot, Theorem 3.1 asserts that the polynomials Δ_{R_a} and Δ_{R_b} are strongly coprime for $a \neq b$.

Thus, we have an explicit set of knots fulfilling the conditions of [CHL10, Theorem 7.7]. By applying the theorem, we get the following linearly independent sets in $\mathcal{F}_n/\mathcal{F}_{n.5}$:

Theorem 5.1. *Let J_0, J_1, \dots be a sequence of knots with vanishing Arf-invariant and linearly independent ρ_0 -invariants. Let $A = \{(a_{i,1}, \dots, a_{i,n})\}$ be a collection of n -tuples of positive integers such that for all $i \neq j$ and $1 \leq m \leq n$, we have $a_{i,m} \neq a_{j,m}$. Given $a = (a_{i,1}, \dots, a_{i,n}) \in A$, let \mathcal{R}_a be the iterated doubling operator $R_\eta^{a_{i,n}} \circ \dots \circ R_\eta^{a_{i,1}}$, where the $R_\eta^{a_{i,m}}$ are the robust doubling operators defined above. Then the set*

$$\{\mathcal{R}_a(J_j) \mid a \in A, 1 \leq j < \infty\}.$$

is linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$.

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